Emergence of synchronization induced by the interplay between two prisoner's dilemma games with volunteering in small-world networks

Yong Chen (陈勇),[1,2,*] Shao-Meng Qin (秦绍萌),[2] Lianchun Yu (俞连春),[2]
and Shengli Zhang (张胜利)[1]

[1]*Research Center for Science, Xi'an Jiaotong University, Xi'an 710049, China*
[2]*Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China*

We studied synchronization between prisoner's dilemma games with voluntary participation in two Newman-Watts small-world networks. It was found that there are three kinds of synchronization: partial phase synchronization, total phase synchronization, and complete synchronization, for varied coupling factors. Besides, two games can reach complete synchronization for the large enough coupling factor. We also discussed the effect of the coupling factor on the amplitude of oscillation of *cooperator* density.

There has been a long history of studying game theory. Some restricted version of the Nash equilibrium concept as early as 1838 was used by the French economist Augustin Cournot to solve the quantity choice problem under duopoly [1]. Since the comprehensive seminal book of Neumann and Morgenstern, game theory has become a powerful framework to investigate evolutionary fate of individual traits under differing competition [2]. In recent years, it has been applied successfully to problems in biology [3], psychology, computer science, operation research, political science, military strategies, economics, and so on [4–6].

The prisoner's dilemma game (PDG) stands as a paradigm of a system which is capable of displaying both cooperative and competitive behaviors through pairwise interactions. After the PDG was first applied by Axelrod on a lattice [7], spatial prisoners' dilemma games (SPDGs) have been studied in various kinds of network models. In the general PDG, each of two players has two strategies, *cooperator* (C) or *defector* (D). If both of them choose the $C(D)$, the player will get payoff $R(P)$. When the D betrays the C, the D will win the income T and the C gets S. Four elements satisfy the order ranking $T > R > P > S$ and usually the additional constraint $(T+S) < 2R$ in repeated interaction. So, the mutual cooperation leads to the highest return for the community and defection is the optimal decision regardless of the other player. In SPDG, the players are located on the nodes. Each player updates his strategy at discrete time steps and has a probability of mimicking his neighborhood strategies.

Szabó *et al.* [8] developed SPDG with voluntary participation, in which players take one of the three strategies, C, D, or *loner* (L). In the traditional PDG, every player participates in this game compulsorily. However, players might drop off risky social enterprise. They do not participate in the game temporarily and earn a smaller payoff σ ($0 < \sigma < R$) on their individual efforts. L is better than a pair of D, but is less than two C. If one of the two players chooses L, the other player is forced to choose L. The payoff is determined by the matrix in Table I.

The purpose of this paper is to describe some interesting peculiarities between two local stock markets or two stocks

in one market by the model of PDG with voluntary participation and Newman-Watts small-world (NWSW) networks [9]. The stock trading can be regarded as a prisoner's dilemma game. Shareholders gain if everyone chooses to cooperate. If any large shareholders choose to sell, the remaining shareholders will likely lose. NWSW can mimic the properties of social networks [9].

There are economic phenomena that one local stock market follows the fluctuations of other stock markets and occasional synchronous events happen among the individual stocks. The correlation between different stock markets has a pivotal role in value-at-risk measures, optimal portfolio weights, hedge rations, and so on [10]. The synchronous events among the individual stocks are also important to the stock market asymmetry [11]. In Ref. [11], Donangelo believes that the synchronous events are caused by the *fear factor*. However, why do shareholders not feel excited together when there are big chances to obtain a profit? Generally speaking, this phenomenon is induced by the interactions among shareholders of different stocks and in different domestic financial markets. In our opinion, the synchronization is caused by conformity. It means that when we find ourselves in the minority in a group, we could change our decision to avoid discomfort caused by that situation. From the social psychology point of view, we understand the fact that conformity is pervasively applicable [12].

Our model is defined as follows. We consider two NWSW networks with the same size and the same rewired bound p but different detailed initial structures. The NWSW network is obtained in practice by the following procedures: (i) Starting with a two-dimensional lattice network with periodic boundary, each player is located on the node with four neighbors. (ii) The long range links are randomly rewired with a certain fraction p.

TABLE I. The payoff matrix of spatial prisoners' dilemma game with voluntary participation.

Player 1 or player 2	C	D	L
C	$R \backslash R$	$S \backslash T$	$\sigma \backslash \sigma$
D	$T \backslash S$	$P \backslash P$	$\sigma \backslash \sigma$
L	$\sigma \backslash \sigma$	$\sigma \backslash \sigma$	$\sigma \backslash \sigma$

*ychen@lzu.edu.cn

Every node in the networks is an agent in the game. Each agent is a pure strategist and can only take one of the three strategies (C, D, L). In each simulation time step, all agents play the PDG with their neighborhoods. The parameters in the payoff matrix are $R=1$, $S=P=0$, $T=b$ $(1.0<b<2.0)$, and $\sigma=0.3$, where b is regarded as temptation. At the next time step, the ith player changes its strategy by following one of its neighborhoods that is selected with equal probability. The probability of this change is defined as

$$W = \frac{1}{1 + \exp[-(E_j - E_i)/\kappa]}, \tag{1}$$

where E_i, E_j are the ith and jth players' total payoffs at the previous round. $\kappa=0.1$ denotes the noise to permit irrational choices. The payoffs of the players will not be counted in the next round. In the case of $\kappa=0$, the strategy of player j is always adopted provided $E_j > E_i$. We found that κ does not have an obvious effect in our model if it is within realistic limits [8]. So, in our model, we only have two parameters, the rewired fraction p and the temptation b.

In order to model the conformity we make an assumption that agents could be affected by the other network. The point of our assumption that the network could not affect its agents is that people usually cannot obtain the perfect information from their group and their neighborhoods will affect their judgment of the global situation. We define a coupling factor F to actualize this assumption. Now, we begin by describing two interplayed PDGs in NWSW networks with the following modifications: (i) At each step, each agent chooses strategy C with a probability $F \times C_{other}$, where C_{other} is the C density in the other network. (ii) The agent will search for a better strategy according to the rule mentioned above.

Clearly, the larger F, the easier one agent is influenced by the agents in the other network. For $F=0$, all agents play PDGs in the network without interplay. A similar model that the players in a PDG are influenced by external constraints has been studied by Szabó [13].

Works by Szabó, Hauert, and Wu reported a comprehensive study of this model in the case of $F=0$ [8,14,15]. One of the most important characters of this model is the persistent global periodical oscillations of three different strategies. If most of the players select C, D will be more profitable; however, when D is dominative, L will make a steady income; after L alleviates the threat of D, C attracts the L to converse. So, three strategies implement a rock-scissors-paper–type cyclic dominance. However, the periodical oscillations of strategies in two different NWSW networks do not show the same amplitude and phase (see Fig. 1). The reason for this difference is that two network structures are not exactly identical and the evolution of SPDG is random.

To measure the differences (or synchronization) between PDGs in two NWSW networks, one conspicuous parameter is ΔC. It is defined as

$$\Delta C = \frac{1}{N} \sum_{t=0}^{N} |c_1(t) - c_2(t)|, \tag{2}$$

where $c_1(t)$ and $c_2(t)$ are the C density at step t in two networks. Clearly, $\Delta C=0$ means the complete synchronization
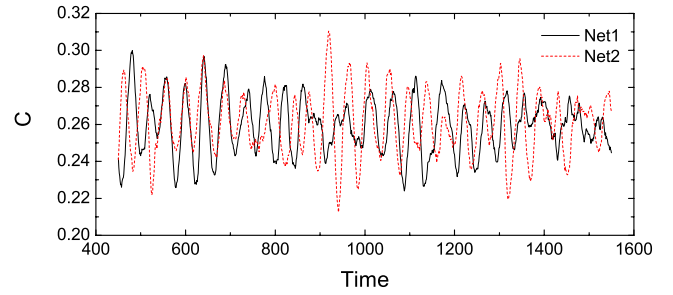

FIG. 1. (Color online) The oscillations of C strategy in two different NWSW networks. Parameters are $b=1.6$ and $p=0.01$.

state. Because there exists the global period oscillation of three strategies in this model, one can define the phase of strategy C in every time step. In this paper, we use Definition B in Ref. [16], and we use the crossing of mean C density as the beginning of the cycle. To study the phase of oscillation, one sets $\Delta\varphi$ as

$$\Delta\varphi = \frac{1}{N} \sum_{t=0}^{N} |\varphi_1(t) - \varphi_2(t)|, \tag{3}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are the phases of strategy C at step t in two networks. $\Delta\varphi=0$ indicates there are no phase differences between PDGs in two NWSW networks. The maximal phase difference is π and $\Delta\varphi=\pi/2$ means no phase synchronization. However, when two networks reach a synchronization state with constant phase difference $\varphi=\pi/2$, we cannot distinguish synchronization from no synchronization. To avoid this puzzle, another parameter Q defined by Kuramoto [17] is introduced:

$$Q = \left| \frac{1}{N} \sum_{t=0}^{N} e^{i[\varphi_1(t) - \varphi_2(t)]} \right|. \tag{4}$$

In the case of phase synchronization, all vectors in the complex space of $e^{i[\varphi_1(t) - \varphi_2(t)]}$ have the same direction, and Q is close to 1. $Q=1$ means total synchronization and $0<Q<1$ means partial synchronization, while for $Q=0$ there is no synchronization at all.

Figure 2 shows the three synchronization parameters depending on p for PDGs in two NWSW networks with various b for $F=0$. Considering that strategy density in this model is periodical oscillation, after the initial transient state, we recorded 25 000 steps as a sampling to calculate these parameters. And, the behaviors of this model will be affected by the size of the network and random seed. All simulations in this paper are performed in networks with 200×200 players and random initial states with an equal fraction of three strategies. All results presented in this paper are the average of 20 trials with different random seed. Obviously, $\Delta\varphi$ fluctuates around $\pi/2$, and Q is close to 0. This clearly demonstrates that the synchronization state does not exist. For $p \leq 0.2$ or $b < 1.4$, ΔC is very small. For $1.5 \leq b \leq 1.9$, ΔC increases monotonically with p. As to $b > 1.9$, the oscillations of strategies become large enough to inevitably lead to
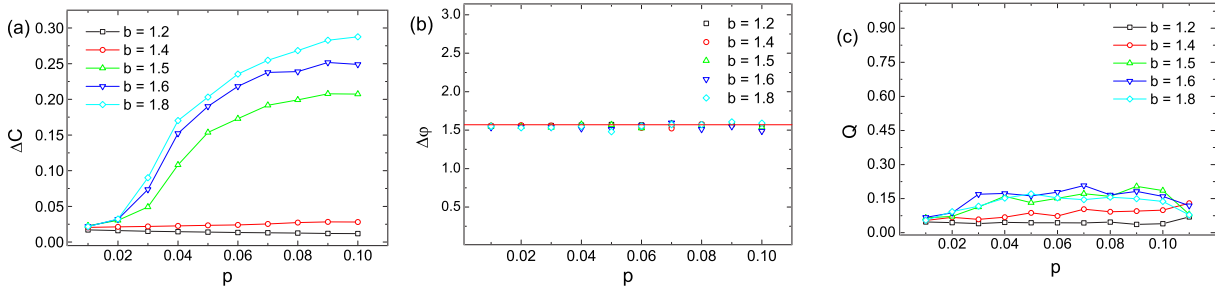
FIG. 2. (Color online) The differences of C strategy between two PDGs without interplay ($F=0$). The red solid line in (b) is $\Delta\varphi=\pi/2$.

the extinction of one strategy. The amplitude of oscillation depends on p and temptation b is conformed by Szabó and Hauert [8].

In Fig. 3, we plot how three parameters ΔC, $\Delta\varphi$, and Q vary with the coupling factor F. Clearly, it shows that the synchronization state exists when F is large enough. For small b $b \leq 1.3$, ΔC is very close to 0 and it looks like an identical synchronization with different phases. For larger b, by enhancing F enough, the networks should reach a synchronization with $\Delta\varphi \approx 0$ and $Q \approx 1$.

It is interesting that three parameters do not always increase or decrease monotonously with F. For example, when $b=1.8$ and $p=0.05$, all three parameters increase up to the first peak at $F=0.004$ which indicates that two networks are not independent and there is a phase discrepancy between them. It means that a partial phase synchronization emerges between two networks. The largest phase discrepancy in this stage is close to π. As F increases near 0.014, three parameters decrease to their level at $F=0$ and the partial phase synchronization is broken. There exists a gap in Figs. 3(c),
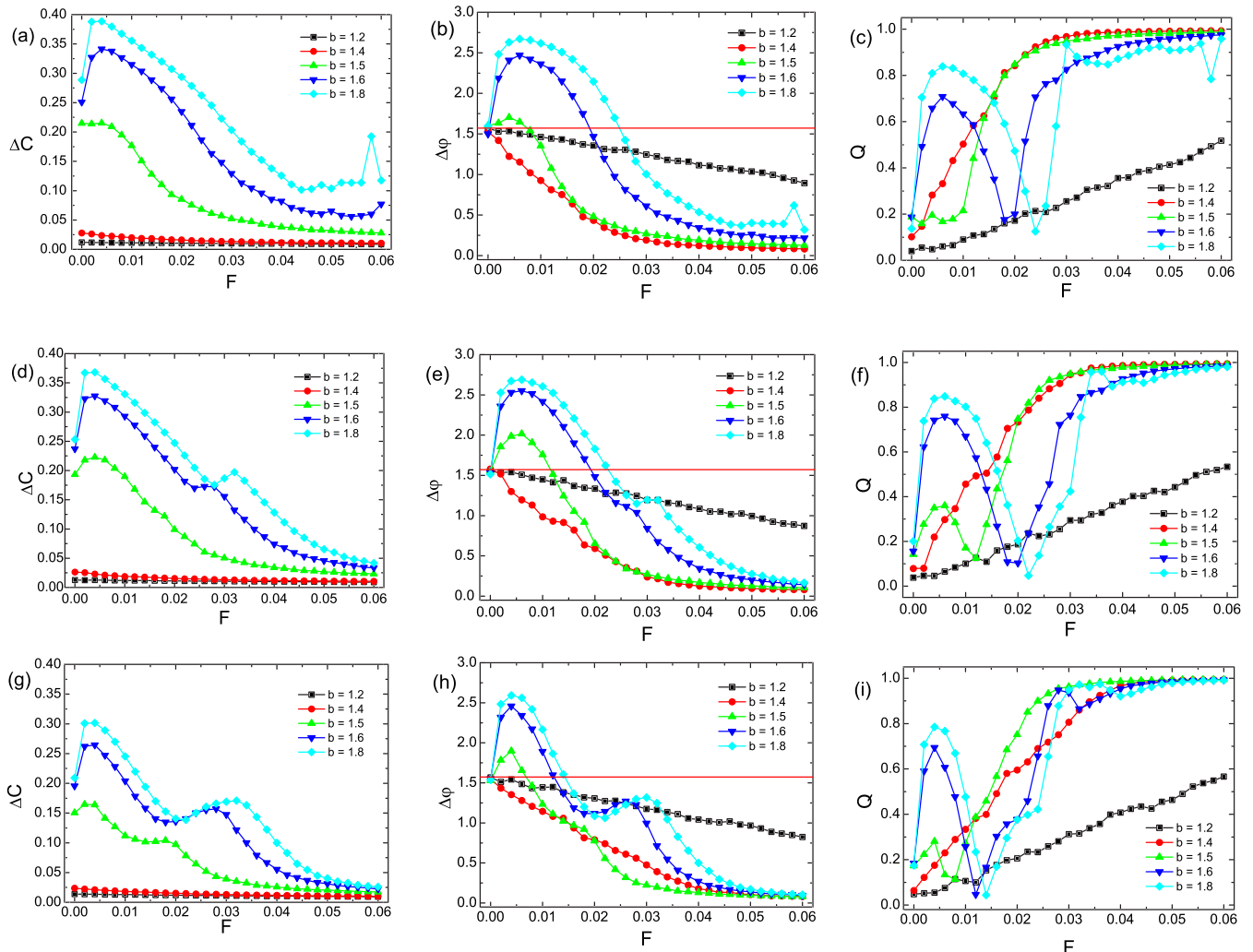

FIG. 3. (Color online) From top to the bottom $p=0.1$, 0.075, and 0.05. The red solid line is $\Delta\varphi=\pi/2$.

3(f), and 3(i) at $F=0.014$. This gap (the first gap) is the boundary between partial and total phase synchronization. Until the F reaches the range $0.03 < F < 0.04$, two networks become correlative once again to reach a total phase synchronization. In this range, ΔC and $\Delta \varphi$ have an abnormal bump. It denotes an augmenting phase discrepancy in phase synchronization. After this slight improvement of coupling, two networks reach complete synchronization. However, we find that Q has a small gap (the second gap) at $F=0.04$. This gap becomes the boundary between complete and phase synchronization. As the above discussion, an interesting phenomenon is that the effect of F is not continuous. In order to achieve a new synchronization, the old one should be broken down first.

ΔC and $\Delta \varphi$ increase with the structural parameters p and b in the partial synchronization stage. p and b are important to the position of the first gap. The larger the p and b, the larger the F for the first gap position. Moreover, Q is closer to 1 for larger p and b with the same F on the partial synchronization. It can be seen from Fig. 3 that the increase of p makes the process of synchronization more difficult. The abnormal bump of ΔC and $\Delta \varphi$ and the second gap disappear gradually on larger p and smaller b. Three different kinds of synchronization can only be observed in the case of large b and small p. Since the large enough amplitude of oscillation inevitably leads to the extinction of one strategy, the synchronization will become unstable for large p and b.

Figure 4 presents how the coupling F affects the amplitude of oscillation of C density. It was found that at the partial phase synchronization stage the amplitude decreases with the coupling factor F. In the regime where the $\Delta \varphi$ has the abnormal bump, the amplitude increases slightly. Then, the amplitude increases monotonously with larger F. Comparing the amplitude with $\Delta \varphi$, it is easy to find that the behaviors of $\Delta \varphi$ and amplitude with F are contrary. It is conjectured that there is a positive feedback for the interplay between games in two networks.
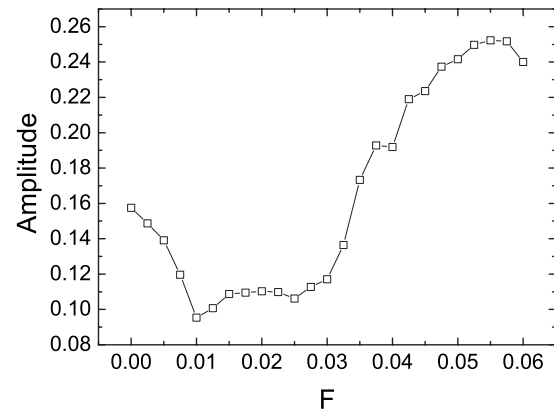

FIG. 4. The amplitude of the oscillation of C density for varied coupling F. Parameters are $b=1.6$ and $p=0.05$.

In this work, we discussed the effect of the interplay between two prisoner's dilemma games with volunteering in NWSW networks. By defining the coupling factor F between two different networks based on the conformity psychology, it was found that the large enough F will lead to synchronization between two networks. We concluded that this model captures the synchronization characteristic in stock markets. To measure the detailed information of synchronization, we introduced three parameters, $\Delta \varphi$, ΔC, and Q. It shows that there are three different kinds of synchronization for different F from our extensive simulations. The network structure p and the temptation b play a very important role on the synchronization between two networks.

[1] J. Nash, Ann. Math. **54**, 286 (1951).
[2] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* (Princeton University Press, Princeton, NJ, 1944); E. Feehr and U. Fischbacher, Econom. J. **112**, 478 (2002); K. Clark and M. Sefton, *ibid.* **111**, 51 (2001).
[3] P. E. Turner and L. Chao, Nature (London) **398**, 441 (1999).
[4] J. M. Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, England, 1982).
[5] J. Hofbauer and K. Sigmund, *Evolutionary Games and Popularion Dynamics* (Cambridge University Press, Cambridge, England, 1998).
[6] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).
[7] R. Axelrod, *Evolution of Cooperation* (Basic Books, New York, 1984); M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992); Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 35 (1993).
[8] G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, 118101 (2002); Phys. Rev. E **66**, 062903 (2002); G. Szabó and J. Vukov, *ibid.* **69**, 036107 (2004).

[9] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[10] M. Martens and S.-H. Poon, J. Bank. Finance **25**, 1805 (2001); C. B. Erb, C. R. Harvey, and T. E. Viskanta, Financ. Anal. J. **50**, 32 (1994); K. F. Kroner and V. K. Ng, Rev. Financ. Stud. **11**, 817 (1998).
[11] R. Donangelo, M. H. Jensen, I. Simonsen, and K. Sneppen, J. Stat. Mech.: Theory Exp. (2006) L11001.
[12] D. G. Myers, *Social Psychology* (McGraw-Hill, New York, 1993).
[13] G. Szabó, T. Antal, P. Szabó, and M. Droz, Phys. Rev. E **62**, 1095 (2000).
[14] C. Hauert, Am. J. Phys. **73**, 405 (2005).
[15] Z.-X. Wu, X.-J. Xu, Y. Chen, and Y.-H. Wang, Phys. Rev. E **71**, 037103 (2005).
[16] J. A. Freund, L. Schimansky-Geier, and P. Hänggi, Chaos **13**, 225 (2003).
[17] Y. Kuramoto and I. Nishikava, J. Stat. Phys. **49**, 569 (1987).